Tuklás Matemática

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Dead Ends of Integers

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The results in this article can be found in an e-print article by Sunic Zoran [1].

Have you ever gone on a road trip to somewhere far and then reached a dead end? Since it is the end of the road, the only way to move further is to move back and look for some other way, right? The same holds if you get to a dead end of the integers. Yes, you read that right; integers have dead ends and that is what you are about to discover in this article.

Consider the set $\mathbb{Z}$ of integers. To begin the search for the dead ends, we first choose two *relatively prime* positive integers; that is, we choose two integers whose greatest common factor is 1. These will serve as our *generators*. For example, let us choose 3 and 5 as our generators. With this generating set, we will be able to write any integer n in the form $n = 3a + 5b$ for some integers a and b. We now define the *length* of an integer (with respect to the generating set) as the least possible value of $|a| + |b|$. For instance, we can write 7 as $3(4) + 5(-1)$ or $3(-1) + 5(2)$ where $|a| + |b| = 5$ for the first one and $|a| + |b| = 3$ for the second one. Since 3 is the least possible value for $|a| + |b|$, 7 has length 3. From now on, whenever we write an integer in terms of the generators, we will use the expression with the least value for $|a| + |b|$.

Now, we will represent the integers graphically with its so-called *Cayley graph*. The vertices would be the integers and an edge between two vertices will be drawn if they differ by 3 or by 5. So starting with 0 which is of length 0, edges will be drawn to 3, -3, 5 and -5, which are the integers of length 1. Next, edges will be drawn from 3 to $3 + 3$, $3 - 3$, $3 + 5$ and $3 - 5$. Similarly, we draw edges from -3, 5, and -5 and in this way, we get the integers of length 2. Continuing further, we obtain the Cayley graph below showing the integers of length up to 5.

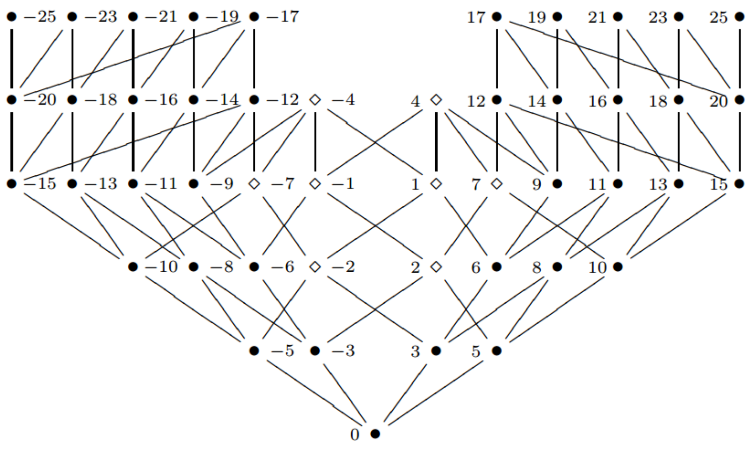

Figure 1: The Cayley graph of $\mathbb{Z}$ with respect to $\{3, 5\}$ (Source: [1])

Consider 12 and 4 that are both of length 4 in this Cayley graph. With 12, adding 3, -3, or -5 will give us integers of length 3 as seen in the graph while adding 5 will give us 17, which is of length 5. So we are able to "move forward" from an element of length 4 to an element of length 5. However, this is not the case with 4. Adding 3, -3, 5, or -5 to 4 will give us elements of length 3 and so we cannot "move forward." Thus, 4 is what we call a *dead end*. Since this also holds for -4 as shown in the graph, -4 is also a dead end. With these, we can then define a dead end of the integers as an integer for which adding a generator gives an integer of the same or lower length, or alternatively, an integer for which any path to that integer in the Cayley graph cannot be extended.

It is easy to see the dead ends of the Cayley graph with respect to the generating set $\{3, 5\}$ since the dead ends are only of length 4; 4 and -4 are in fact the only dead ends as a consequence of the results later. However, different generating sets produce different dead ends. So dead ends with respect to larger generators would be difficult to determine since it will not be simple to draw the Cayley graph. So we ask, given any generating set consisting of 2 integers, how do we find all the dead ends with respect to that generating set?

To answer this, we classify the integers into three. Suppose our generators are x and y. So we can write any integer as $n = xa + yb$. An integer is *positively generated* with respect to the generating set if both a and b are non-negative. An integer is *negatively generated* with respect to the generating set if both a and b are non-positive. If an integer is neither positively generated nor negatively generated, we call that integer a *Frobenius value*. If our generators are 3 and 5, 13 is positively generated since $13 = 3(1)+5(2)$. On the other hand, -12 is negatively generated since $-12 = 3(-4)$. Lastly, 2 is a Frobenius value because it is neither positively nor negatively generated since $2 = 3(-1) + 5(1)$.

From this, note that if an integer is positively generated, we can obviously always extend the path to that integer by adding a positive generator. Similarly, if an integer is negatively generated, we can always "move forward" to an element of larger length by adding the negative of a generator. Thus, our only candidates for the dead ends would be the Frobenius values.

Now, knowing the largest Frobenius value would be of great help for our search of the dead ends. But that is not a problem because given two integers c and d, there is in fact a formula for finding the *Frobenius number*, the largest Frobenius value. It is given by $cd - c - d$. Moreover, note that if an integer n is a Frobenius value, then $-n$ is also a Frobenius value (and n and $-n$ would be of same length).

Again, if we are using the generators 3 and 5, the Frobenius number is $(3)(5) - 3 - 5$ or 7. Manually searching for other Frobenius values, we get 1, 2, 4, 7, and their negative counterparts. Now which of these would be the dead ends? Referring to the graph, note that ± 1 has length 3, ± 2 has length 2, ± 4 has length 4, and ± 7 has length 3. But as we said earlier, only 4 (and -4) are the dead ends. Notice that 4 and -4 are the Frobenius values with the largest length! We then conclude and we can prove (which you may try by using some arguments based on the definition of Frobenius values and maximality of length) that, in fact, **the dead ends of the Cayley graph of integers with respect to a generating set are the Frobenius values of maximal length with respect to that generating set.**

We now know that the dead ends are just the Frobenius values of maximal length. However, looking for the Frobenius values and checking for the lengths can be tedious especially if the generators are large. But that is not a problem anymore because after lots of "trial and error," or better yet, "guess and check" and several attempts to formulate a closed formula, there is a solution for all the dead ends given a generating set. If we choose two positive relatively prime integers x and y with $x > y \geq 1$ as the generators, the dead ends of integers (or the Frobenius values of maximal length) with respect to this generating set are given as follows:

- If $x+y$ is even, then there are exactly $y-1$ dead ends, they all have length $(x+y)/2$, and they are given by
$$d = \frac{(x+y)(2\alpha - y)}{2}$$
for $\alpha = 1, 2, ..., y-1$
- If $x+y$ is odd, then there are exactly $2(y-1)$ dead ends, they all have length $(x+y-1)/2$, and they are given by
$$d = \frac{(x+y)(2\alpha - y) \pm y}{2}$$
for $\alpha = 1, 2, ..., y-1$

So there, we finally know how to determine the dead ends of integers with respect to 2 generators. But how about if we use 3 generators? or 4? or more? Some results have already been found for 3 generators but instead of presenting them here, you will have more fun finding these out for yourself.

If you are interested in knowing more, you might want to study Group Theory to find more interesting "groups" with more complicated dead ends. Don't let this article be the dead end of your interest for the "real" algebra; mathematics, in contrast to this article, will never have a dead end. Or even if you think you have reached one, you can always move back a little, and find other ways.

Reference

[1] Sunic, Zoran. "Frobenius Problem and Dead Ends in Integers." Cornell University Library (2006). Web. 04 Nov. 2014. (`http://arxiv.org/abs/math/0612271`).

About the author

Luis S. Silvestre Jr. obtained his B.S. in Mathematics at the Ateneo de Manila University in 2015 and is currently taking his Master's degree at the same university.

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Tuklás Matemátika is the online journal of the Ateneo Mathematics Department. Aimed at exposing high school students and the general public to topics beyond usual high school curricula, Tuklás features informative and interesting articles about people and ideas from different areas of mathematics. Tuklás Matemátika originally served as an online supplementary journal for the PEM.

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