Funny Expectations

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Posted October 23, 2016 at www.ateneo.edu/ls/sose/mathematics/tuklas

If a certain variable X has possible values $x_1, x_2, \cdots x_n$ with respective probabilities $p_1, p_2, \cdots p_n$, we can come up with a quantity that can roughly represent X based on its possible values. An intuitive way of establishing this quantity is by taking the weighted sum of the x_i's with respect to their corresponding probabilities. This quantity is called the **expected value** or the **expectation** of the random variable X and is denoted by $E(X)$. Think that X is called a random variable since its value is indeed probabilistic. In this case, $E(X) = x_1p_1 + x_2p_2 + \cdots + x_np_n$. Of course, we can extend this to a case wherein X can a achieve an infinite number of possible values.

For example, if we let X denote the outcome of rolling a fair dice, then X can be $1, 2, 3, 4, 5$, or 6. Since the dice is fair, each outcome has probability $\frac{1}{6}$. Thus, $E(X) = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = 3.5$. The expectation is not called "expectation" for nothing. Suppose we have another dice with some deformities that cause it to turn up a 6 more often, then the next time we roll this dice, we must be **expecting** a 6 more than any other outcome. To be specific, suppose the dice turns up a 6 with probability $\frac{5}{6}$ and turns up each of the five other possible outcomes with probability $\frac{1}{30}$. Let Y denote the number rolled using this biased dice, then $E(Y) = 1(\frac{1}{30}) + 2(\frac{1}{30}) + 3(\frac{1}{30}) + 4(\frac{1}{30}) + 5(\frac{1}{30}) + 6(\frac{5}{6}) = 5.5$. Notice how close this value is to 6.

A notable significance of the expectation is its thorough use in making strategies, particularly in decision-making. Consider a game wherein a fair coin is tossed; if the coin lands on a head, we receive twenty pesos and when it lands on a tail, we receive only five pesos. A good question would be "How much should we be willing to bet to play the game?" This question needs to be asked because, naturally, no person is generous enough to offer the game for free unless this person is a philanthropist. At the same time, we would be considered foolish if we were willing to pay thirty pesos to play the game.

So what we can do is to decide based on expectations. Since the game just gives us money based on a result of a fair coin, then we receive five pesos with probability $\frac{1}{2}$ and receive twenty pesos with probability $\frac{1}{2}$. Hence, the expected amount of money we receive from the game is $5(0.5) + 20(0.5) = 12.5$. Then we can decide this way: Play the game if the game charges us at most 12.5 pesos and decline to play if it charges us more than 12.5 pesos.

Now, let us consider the popular sweepstakes Lotto game we have in the country. A few days ago, the jackpot prize for the 6/49 lottery was Php 23563096.00. We can enter our 6-digit combination to the lottery by paying 10 pesos. The probability that our combination wins is $\frac{1}{49C6} = \frac{1}{13983816}$. If our combination wins, then we will be instant millionaires; but if we lose, we will get nothing. Many people consider this game a fair deal because ten pesos is very minimal compared to the jackpot prize. Meanwhile, others do not even think about playing because the chance of winning is just too slim. How about we use expectations? In our example, our expected winnings will be $23563096(\frac{1}{13983816}) = 0.0889$. Since the expected value is much less than what we need to pay to enter the lottery, we might be better off waiting for the next lottery instead and hoping that the jackpot prize increases even more.

Using expectations as a strategy in making decisions seems very useful. But these expectations can be a bit funny and confusing sometimes. Consider another game that also involves tossing a fair coin. This time, the coin is tossed multiple times until a head appears. If n is the number of tosses it takes to get the first head, we win 2^n pesos. That is, if the first head occurs in the first toss, you win 2 pesos; if the first head occurs in the second toss, you win 4 pesos; if the first head occurs in the third pesos, we win 8 pesos and so on. Seems like a fun game, but the same question arises: "At most how much should we be willing to pay to enter the game and call it a good deal?"

So let us stick to our game plan and use expectations once again. In this game, we win 2 pesos with probability $\frac{1}{2}$. For us to win four pesos, the first toss must be a tail and the second toss must be a head, so that has probability $\frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$. For us to win eight pesos, we need tails in the first two tosses and a head on the third toss and that has probability $(\frac{1}{2})^3 = \frac{1}{8}$. In general, we win 2^n pesos with probability $\frac{1}{2^n}$. So if we let X be our prize money in this game, then

$$\begin{aligned} E(X) &= 2\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) + 8\left(\frac{1}{8}\right) + \cdots \\ &= 1 + 1 + 1 + \cdots \\ &= \infty \end{aligned}$$

Is there a problem? Well, if the person who offers this game charges us two pesos to play it, then there is no problem since $2 < \infty$,. Similarly, if he offers it for 10 pesos, then we go ahead and play the game, since $10 < \infty$. If he offers it for 1000 pesos, then we still should play since $1000 < \infty$ so the joke's on him. But this list continues. The calculated expectation leads us to conclude that no matter how expensive the game is, we must accept the offer to play it! If you don't see what's funny about this, then think of it this way: Based on the expectation, it is logical to pay a billion pesos to play the game because a billion is certainly less than infinity. But suppose that on the first turn, the coin landed heads, then you will win two pesos and will be certainly in the news headlines the very next day: "Billionaire wastes money on silly game, goes home with two pesos".

This problem is called the "St. Petersburg Paradox" and has puzzled the mathematician Daniel Bernoulli. This problem actually caused a debate on whether the use of expectations should be rejected in decision making. The paradox is eventually resolved by Bernoulli by saying that expectation should not be based on how much money is won, but should be based on the utility of money with respect to the person playing the game. Thus, the amount a billionaire can be willing to pay to play the game varies with the amount a pauper can be willing to pay. To make things clearer, 20 pesos is much more valuable to a person who has not eaten for days than to a person like Manny Pacquiao. The former can not give up a lot to play the game, but the latter can give up a larger amount of money to play it.

It is hard to come up with a solution like Bernoulli's. So here is another funny expectation that you can think about: Your *Ninong* presents to you two envelopes each containing money. One envelope has money twice the amount of what's inside the other envelope. He gives you the chance to pick one envelope at random; the other envelope goes to his other godchild. After picking an envelope, your *Ninong* tells you that you have the chance to swap envelopes only if you haven't opened them yet. Quick question: "Should you swap?"

Okay, let us analyze this. Suppose that the money in the first envelope you get is X pesos. Then since you are told that one envelope contains twice the amount of the other, then that means that the other envelope contains either $\frac{X}{2}$ pesos or $2X$ pesos. So we create our strategy based on expectations. We let Y be the amount of money in the other envelope. Therefore $E(Y) = \frac{1}{2}(\frac{X}{2}) + \frac{1}{2}(2X) = \frac{X}{4} + X = \frac{5X}{4}$. Since $\frac{5X}{4}$ is definitely bigger than X, the amount you have in your envelope, then you swap. So after swapping, you get the other envelope. This envelope contains Z pesos. But wait, the other envelope must be twice what you have or is half of what you have, each with probability $\frac{1}{2}$. The expected amount in that envelope must be $\frac{1}{2}(\frac{Z}{2}) + \frac{1}{2}(2Z) = \frac{Z}{4} + Z = \frac{5Z}{4}$. So since $\frac{5Z}{4}$ is bigger than Z, you ask your Ninong to swap envelopes again. Then you calculate the expectation again, you compare and then swap again. Calculate expectation, compare, then swap. Calculate expectation, compare, then swap. Expectation, compare, swap. You do this over and over again until your Ninong gets so tired that he decides to give both envelopes to his other godson instead. You go home with nothing.

Let us resolve this. At the very first instance the envelopes were presented, you have no hint about which envelope has the larger amount of money. But after choosing the first envelope, and given the chance to swap, all of a sudden you are able to use expectations to decide which one has more money. What happened? The answer is that after the choice of the first envelope, we assumed that the second envelope could contain an amount that could never actually be its value. For example, Envelope A has 4 pesos while Envelope B has 8 pesos; you choose Envelope A at random and conclude that Envelope B has either 8 pesos or 2 pesos; but B cannot have 2 pesos! From the very start, the sum of the money in the two envelopes is constant. By admitting the possibility that Envelope B can have 2 pesos, then by the same reasoning, Envelope A can have either 1 peso or 4 pesos. But in either case, the sum of the money in the envelopes would have become 3 pesos or 6 pesos. You were merely swapping envelopes and so, the original 12 pesos should not have trimmed down. So to resolve the paradox, we need to ensure that the sum of the money in the two envelopes is constant. To do this, we can assume that the money in one envelope is X and the other has $2X$. If you choose the envelope that has X pesos and then swap, then you will get $2X$ pesos, which is a gain of X pesos. On the other hand, if you chose the envelope that has $2X$ pesos and then swap, then you will get X pesos instead, a loss of X pesos.

Thus your expected gain from swapping is $\frac{1}{2}(X) + \frac{1}{2}(-X) = 0$. Therefore, swapping is not that different from keeping your first envelope. The problem above is called the Exchange Paradox.

These two problems presented are just two of the many paradoxes that question the use of the expectation in decision theory. However, we see that, eventually, these paradoxes can be resolved by careful observation and analysis. Hence, much care is needed when using expectations to avoid paradoxes like these ones. But it must be noted that despite the proper use and calculation of the expectation in making optimal decisions, in Math expectation is still different from reality. Being able to properly use and analyze mathematically does not give you the entitlement to laugh at or mock people who, for example, pay a lot of money for lotto tickets. On the flip side, you can always gain a lot of money through the lottery despite the very low chances of hitting the jackpot, and that sometimes means having to abandon calculated and well-thought-out strategies. Similarly, people can actually make expectations of you but it is not always the case that you must stoop down to these expectations. Instead, it is best that you defy them and even exceed your own.

About the author

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