Tuklás Matemátika

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Mathematical Modelling Using Differential Equations

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Finance, economics, data science, biology, and physics are just some of the fields wherein complex problems and phenomena arise. Most of the time, solving or understanding these require mathematical tools and strategies. The facets of the problem or phenomenon are translated to the mathematical language to create and develop a mathematical model from which conclusions can be derived. In this article, we focus on a mathematical model in the field of biology, specifically epidemiology—the study of the spread and control of diseases.

Due to the ethical and logistical issues, performing and controlling scientific experiments to understand human diseases are often difficult. In these cases, mathematical modelling becomes an important tool for understanding the spread of disease and formulating control measures.

A mathematical model usually starts simple, with restrictive assumptions, in order to make the analysis more manageable. The assumptions often dictate how the model can be solved or understood and, in turn, how much information the model can provide. As mathematical methods improve, the assumptions of the model are relaxed so that it is closer to reality.

One important class of mathematical models are those that make use of **differential equations**. These are equations that involve mathematical quantities called **derivatives**, which are usually used to represent the rate of change of a quantity x with respect to a time t. It is denoted by $\frac{dx}{dt}$ or $x'(t)$. One simple interpretation of the derivative is the following: If $x'(t)$ is negative, then x is decreasing at time t; on the other hand, if $x'(t)$ is positive, x is increasing at t. If $x'(t) = 0$ for all values of t, then x is a constant quantity. When multiple variables are involved in a problem, a **system of differential equations** is used and solved.

The following is an example of a mathematical model (from [1]) involving a system of differential equations:

$$S'_H(t) = \mu_H N_H - \lambda_H \frac{I_V(t)}{N_V} S_H(t) - \mu_H S_H(t) \tag{1a}$$

$$I'_H(t) = \lambda_H \frac{I_V(t)}{N_V} S_H(t) - e^{-\mu_H \tau} \lambda_H \frac{I_V(t-\tau)}{N_V} S_H(t-\tau) - \mu_H I_H(t) \tag{1b}$$

$$R'_H(t) = e^{-\mu_H \tau} \lambda_H \frac{I_V(t-\tau)}{N_V} S_H(t-\tau) - \mu_H R_H(t) \tag{1c}$$

$$S'_V(t) = \mu_V N_V - \lambda_V \frac{I_H(t)}{N_H(t)} S_V(t) - \mu_V S_V(t) \tag{1d}$$

$$I'_V(t) = \lambda_V \frac{I_H(t)}{N_H(t)} S_V(t) - \mu_V I_V(t) \tag{1e}$$

In this system, equations (1b) and (1c) are examples of a special type of differential equations called **delay differential equations** (or **DDE**). These are differential equations wherein the present rate of change of a quantity is dependent on past values of one or more quantities. In this example, the expression $t - \tau$ indicates that the rate of change is dependent on values τ units of time ago.

The model (1a)–(1e) represents the spread of dengue fever in a community. The model makes use of two population classes: the humans and the vectors (or the mosquitoes). In the human population, there are three subclasses: the susceptible, the infected, and the recovered. On the other hand, the vector population has two subclasses: the susceptible and the infectious. One human can be infected through an infected vector, while a vector can become infectious through an infected human. Certain assumptions are also presented which are used in the model's analysis. The movements of the populations' members from one subclass to another are summarized in the figure below.

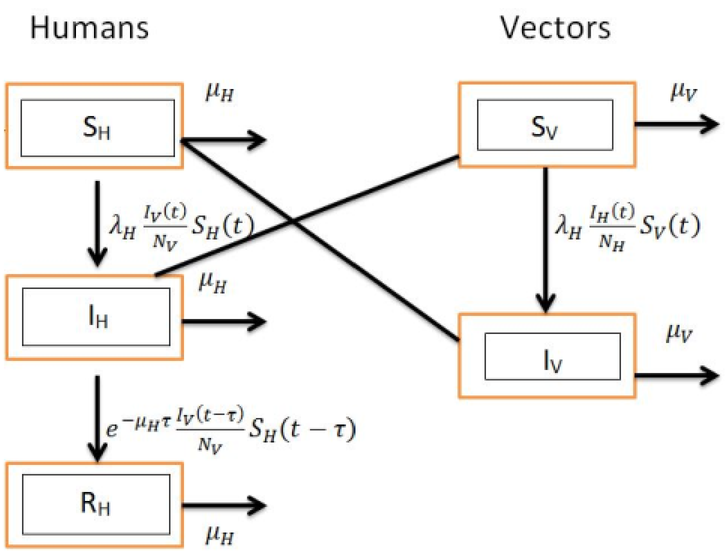

Figure 1: Compartmentalization of the model.

In analyzing the spread of disease using mathematical models, some concepts are of importance.

1. **Fixed points** are constant solutions of the differential equations. Usually, the model has two fixed points: the **disease-free fixed point** in which there is no disease in the community (there are no infected humans and infectious vectors), and the **endemic fixed point** in which the disease stays in the community because there are still infected humans and infected vectors.
2. A **threshold parameter** is a quantity, usually dependent on the parameters of the model, that is the point where the nature of the fixed point(s) changes.
3. The **basic reproductive number** refers to the number of secondary infections an infected individual can produce in a totally susceptible community. This is usually related to the threshold parameter.
4. The fixed points are analyzed using **stability analysis**. A fixed point is **stable** if every solution around it stays "near" the fixed point; moreover, a fixed point is **asymptotically stable** if every solution "approaches" it. A **bifurcation** occurs when a parameter reaches a certain value so that the stability of the fixed point(s) changes: from stable to unstable or vice-versa.
5. **Numerical simulations** are conducted in order to verify conclusions from stability analysis.

These concepts allow mathematicians to make conclusions about the phenomenon or problem that the system of differential equations seeks to model. While the details are too intricate to discuss in this article, we emphasize that the information obtained from such a model can be used to make important steps to reduce the spread of dengue fever.

Mathematical modelling is one of the many ways mathematics is used in diverse fields. Despite many complexities and abstraction, mathematical models can be the source of a breakthrough for solving real-world problems and can shed some light on the unknown.

References

[1] J.P Soto, "A Model For Dengue Fever with Finite Infectious Period Using Delay Differential Equations", Undergraduate Thesis, Mathematics Department, Ateneo de Manila University, 2016.

About the author

John Paolo O. Soto finished his B.S. in Mathematics at the Ateneo de Manila University in 2016 and is now taking his M.S. in Mathematics at the same university.

About Tuklás Matemátika

Tuklás Matemátika is the online journal of the Ateneo Mathematics Department. Aimed at exposing high school students and the general public to topics beyond usual high school curricula, Tuklás features informative and interesting articles about people and ideas from different areas of mathematics. Tuklás Matemátika originally served as an online supplementary journal for the PEM.

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