Mathematics of Physics

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Eugene Wigner, a famous physicist, thought it miraculous that mathematics is very successful in modeling the phenomena considered important by physicists, and by extension, chemists. Today, it is also quite miraculous that physics has helped in the solution of several important mathematical problems, such as the topology of low-dimensional manifolds, whose solution are inspired by the methods of string theory. Mathematics is, however, not as successful in the modeling of biological systems. One reason why mathematics is not as successful in its applications to the biological and social sciences is that these sciences have what is called a hierarchy, which is not captured by the usual differential equations used to model physical phenomena. Physics is reductionist-to study a physical system, one studies its constituent parts, and put the solutions together at the end. On the other hand, the study of an organ cannot be sufficiently represented by the study of its constituent tissues, the study of tissues cannot be broken down in the study of its constituent cells, and so on up and down the line. This remains a challenge for mathematicians.

In the 1660s Isaac Newton invented **calculus** just so he could write his Newton's Laws, more precisely his Second Law. The second law expresses the time rate of change of momentum of a particle, or an extended body, as equal to the total forces acting on the particle or body. The equation, in modern notation, is $F = \frac{dp}{dt}$. The basis of calculus is the concept of **limit**. With the limit concept, one may study the behavior of functions at interesting points. For instance the function $f(x) = \frac{x^2+x-2}{x+2}$ has an interesting behavior at the number $x = -2$. Although f is not defined at $x = -2$, its behavior at numbers near $x = -2$ is that the values of the function are near -3. Consider the function $g(x) = \frac{2}{(x-1)^2}$. For large values of x, either positive or negative, the values of $g(x)$ are near zero while remaining positive. This indicate that the graph of $y = g(x)$ has the x-axis as a horizontal asymptote. Finally, consider the function $c(n) = \frac{n^3-5n+3}{2n^3+n^2}$, whose domain are the positive integers. For large integers n, the behavior of $c(n)$ is that it goes nearer and nearer to the value $\frac{1}{3}$. For the first function, we write $\lim_{x\to-2} \frac{x^2+x-2}{x+2} = -3$, and similarly for the other two examples.

Using his calculus, Newton and his contemporaries was able to solve not only simple and complicated problems of the physics of his day, but also to solve important geometrical problems. Among these, the two most important, are the following: finding tangent lines to curves and finding areas of regions between curves. These are the basic content of calculus. Another important development arising from calculus is the use of power series in expressing functions. A power series is an expression of the form

$$a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots ,$$

where the dots at the end indicate that the sum does not end, while the rest of the terms are of the same pattern as the ones written above, i.e., $a_n(x-x_0)^n$.

Examples of power series, or what are also called Taylor series,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots ,$$

$$e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \cdots ,$$

$$\sin x = x - \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots .$$

One value of series is obviously computational. If one wishes to compute, say, $e = e^1$, then just substitute 1 for x in the second series above to see that up to 4 decimal places, $e = 2.7128$. One can similarly compute $\sin 1$.

Another contribution of physics, now that calculus is available, is to model various physical phenomena, such as the decay of radioactive substances, and if one made a few simplifying assumptions, the growth of a population of bacteria with constant source of nutrients. Other phenomena in the ambit of physics are heat conduction and convection, gravitation, motion of ideal gases, velocities of particles near the speed of light, path of particles under magnetic fields, and numerous others.

Models of all these fall under a vast mathematical field, **differential equations**. A differential equation is an equation involving **derivatives** of functions. Given a function $f(x)$, form another function $\delta f(x,h) = \dfrac{f(x+h) - f(x)}{h}$, where x and $x+h$ are in the domain of $f(x)$. The behavior of this function as h goes nearer and nearer to 0 is called the derivative of f at x, written $f'(x)$ or $\dfrac{df}{dx}$ or $\dfrac{\partial f}{\partial x}$ (if the function f depends on more than one variable). One may interpret it as the instantaneous rate of change of f at x. Geometrically, it may be interpreted as the slope of the tangent line to the curve $y = f(x)$ at the point $(x, f(x))$. Newton's second law above is a differential equation. Others are

$$\frac{df}{dx} = f(x), \quad \frac{d}{dx}\left(\frac{df}{dx}\right) + f(x) = 0, \quad \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Countless books are written tackling the modeling of physical phenomena and solving the differential equations of the models. One method of solution is the use of power series. We see here the interconnectedness of various mathematical topics.

Another gift of physics to mathematics is Fourier series. Joseph Fourier, in the 18th century, provided the first set of recipes in solving differential equations using Fourier series methods. (Of course, the method was not named after him while he was alive.) It arose from his studies of models of heat conduction, specifically, the heat equation. His idea is to express a function $f(x)$ as a series

$$a_0 + a_1 \cos x + a_2 \cos(2x) + a_3 \cos(3x) + \cdots +$$

$$b_1 \sin x + b_2 \sin(2x) + b_3 \sin(3x) + \cdots .$$

The coefficients a_n and b_n are expressions involving the **integral**, the other pillar of calculus. Viz.,

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos(nx) dx,$$

$$b_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) \sin(nx) dx.$$

Finding the Fourier series of the function $f(x) = x$ leads to the astonishing series identity

$$\frac{\pi^2}{\sqrt{6}} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

The definite integral of a function f from a to b is written $\int_a^b f(x)\,dx$ and is defined as the limit $\lim_{n\to\infty} \sum_{i=0}^{n} f(c_i)(x_{i+1}-x_i)$, where $x_i \leq c_i \leq x_{i+1}$ and such that n increases so that the maximum of the lengths $x_{i+1}-x_i$ goes to zero. As the expression $f(c_i)(x_{i+1}-x_i)$ may be read as height times width, the sum in the limit may be interpreted as approximating the area of the region, from a to b, under the curve $y = f(x)$ and above the x-axis, whenever $f(x) \geq 0$. Thus $\int_a^b f(x)\,dx$ may be taken to be the area of such a region.

Another area of mathematics used in physics is that of complex analysis, the study of functions whose domain and range are **complex numbers**. One need only to look at the wave equation of quantum mechanics to see its importance,

$$\frac{i}{\hbar}\frac{\partial \psi}{\partial t} = k^2 \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi(x).$$

Here $\hbar$ is Planck's constant, k is a constant, and i is the complex number such that $i^2 = -1$. Thus, the basic differential equation of quantum mechanics cannot be written without the complex numbers!

Calculus, series, Fourier series, and complex analysis are just a few of the many mathematical topics needed to study physical phenomena. Others are matrices, group theory, probability theory, geometry of curves and surfaces, etc. Physics has provided impetus to all of these. An aspiring mathematician then should not neglect the study of the sciences, as these are the main source of important mathematics.

About the author

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