Tuklás Matemática

Ateneo Mathematics Department

PEM Olympiad 2016

Tuklas Special Issue

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The Program of Excellence in Mathematics (PEM) Olympiad was held last October 29, 2016. During the competition, PEM participants were given three hours to solve three challenging math problems. This Tuklas special issue features these PEM Olympiad problems, together with their solutions.

PEM Olympiad 2016: Level A

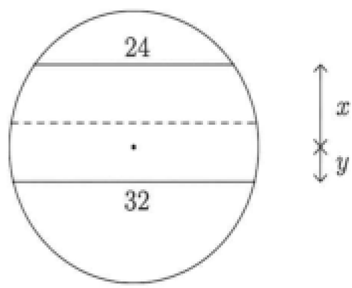
1. Find all primes which can be written both as a sum of two primes and as a difference of two primes. (Brazil, 1988)

   Solution. We claim that the only such prime is 5.
   Let p be such a prime, equal to the sum of two primes and the difference of two primes. Clearly, $p \neq 2$ since 1 is not a prime. Thus, p is odd, so $p = q + 2 = r - 2$ for some primes q and r.
   If $q \equiv 1 \pmod 3$, then $3|p$, so $p = 3$. In this case, $q = 1$, but this is not prime.
   If $q \equiv 2 \pmod 3$, then $p \equiv 1 \pmod 3$ so $3|r$. Thus, $r = 3$.
   Finally, if $q \equiv 0 \pmod 3$, then $q = 3$, so $p = 5$ and $r = 7$. □

2. Two parallel chords of a circle have lengths 24 and 32, respectively, and the distance between them is 14. Determine all possible lengths of another parallel chord midway between the two chords. (Hong Kong, 2014)

   Solution. Let r be the radius of the circle. Denote by x and y the distances from the two chords to the center of the circle (see the figure). Since the perpendicular from the center to a chord bisects this chord, we see by the Pythagorean Theorem that $12^2 + x^2 = r^2$. Similarly, we have $16^2 + y^2 = r^2$. Combining the two equations gives $12^2+x^2 = 16^2+y^2$, which, upon simplification, becomes $(x-y)(x+y) = x^2-y^2 = 112$. There are two possibilities.

   - If the two chords lie on the same half of the circle, then $x - y = 14$, which gives $x + y = 8$ and $y = -3$. This is clearly impossible.
   - If the two chords lie on opposite halves of the circle, then $x + y = 14$, which gives $x - y = 8$. Solving the equations, we get $x = 11$ and $y = 3$. Hence $r = \sqrt{265}$. Thus, the desired chord is at a distance of $\frac{11-3}{2} = 4$ from the center. By the Pythagorean Theorem again, its length is $2 \times \sqrt{(\sqrt{265})^2 - 4^2} = 2\sqrt{249}$. □



3. Let a, b, and c be positive real numbers such that $abc = 1$. Prove that

$$a^3 + b^3 + c^3 + (ab)^3 + (bc)^3 + (ca)^3 \geq 2(a^2b + b^2c + c^2a).$$

Solution. Using the rearrangement inequality on (a, b, c) and (a^2, b^2, c^2), we have

$$a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a.$$

Again, by the rearrangement inequality, but this time on $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$ and $(\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2})$, we have

$$\begin{aligned}
(ab)^3 + (bc)^3 + (ca)^3 &= \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} \\
&\geq \frac{1}{a^2}\frac{1}{c} + \frac{1}{b^2}\frac{1}{a} + \frac{1}{c^2}\frac{1}{b} \\
&= \frac{a}{c} + \frac{b}{a} + \frac{c}{b} \\
&= a^2b + b^2c + c^2a
\end{aligned}$$

Adding two inequalities above gives the desired result. □

PEM Olympiad 2016: Levels B and C

1. The point $P(a, b, c) \in \mathbb{R}^3$ moves to a new point $P'(a', b', c')$ every minute such that
   (a) (a', b', c') is a permutation of (a, b, c); or
   (b) $a' = \sqrt{4bc}$, $b' = c - a - b$, and $c' = b - a - c$.

   If point P starts at $(1, \sqrt{2}, 1)$, determine whether it will eventually reach the point $(\sqrt{10}, -2, -1)$. If yes, what is the least number of minutes it will take to do so?

   Solution. Let S be the sum of the squares of the coordinates of a point. Then operation (a) does not change S. On the other hand, operation (b) does the following:

$$\begin{aligned}
(a')^2 + (b')^2 + (c')^2 &= 4bc + (c - a - b)^2 + (b - a - c)^2 \\
&= 4bc + (a^2 + b^2 + c^2 - 2ac - 2bc + 2ab) \\
&\quad + (a^2 + b^2 + c^2 - 2ab - 2bc + 2ac) \\
&= 2S.
\end{aligned}$$

   Therefore, operation (b) doubles S.
   Since P starts at $(1, \sqrt{2}, 1)$, the current S is 4. The next S when applying operation (b) is 8, then 16, and so on.
   Since $S(\sqrt{10}, -2, -1) = 10 + 4 + 1 = 15$, this cannot be in the sequence of S's above.
   Therefore, the point will never reach $(\sqrt{10}, -2, -1)$. □.

2. A sequence $\{a_1, a_2, \ldots, a_n\}$ of positive integers is said to be a *dragon sequence* if the last digit of a_k is the same as the first digit of a_{k+1} (here $k = 1, 2, ..., n$ and we define $a_{n+1} = a_1$). For example, $\{414\}$, $\{208, 82\}$, and $\{1, 17, 73, 321\}$ are all dragon sequences. At least how many two-digit numbers must be chosen at random to ensure that a dragon sequence can be formed among some of the chosen numbers? (Hong Kong, 2014)

   Solution. If we take any 46 two-digit numbers, then only 44 two-digit numbers are not chosen. Hence, we have one of the 36 pairs $\{\overline{AB}, \overline{BA}\}$ where $1 \leq A < B \leq 9$, or one of the 9 numbers $11, 22, \cdots, 99$. In either case, we get a dragon sequence by definition.
   Next, suppose we only pick those two-digit numbers whose units digit is smaller than its tens digit. There are altogether 45 such numbers. If we get a dragon sequence among these numbers, say $\{\overline{A_1A_2}, \overline{A_2A_3}, ..., \overline{A_nA_1}\}$, then $A_1 > A_2 > ... > A_n >$

A_1, which is a contradiction. So these 45 numbers contain no dragon sequence. It therefore follows that the answer is 46. $\square$

3. Let $\{F_n\}$ be the Fibonacci sequence defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for integers $n \geq 1$. Define $V_n = \sqrt{F_n^2 + F_{n+2}^2}$ for integers $n \geq 1$. Prove that for any integer $n \geq 1$, V_n, V_{n+1}, and V_{n+2} are the sides of a triangle of area 1/2. (Brazil, 1997)

   Solution. Let $P(0, F_{n+2})$, $Q(F_{n+4}, 0)$, and $R(F_{n+3}, F_n)$. Then $PQ = V_{n+2}$,

$$PR = \sqrt{F_{n+3}^2 + (F_{n+2} - F_n)^2} = \sqrt{F_{n+3}^2 + F_{n+1}^2} = V_{n+1}$$

and

$$QR = \sqrt{F_n^2 + (F_{n+3} - F_{n+4})^2} = \sqrt{F_n^2 + F_{n+2}^2} = V_n.$$

It remains to show that $[PQR] = \frac{1}{2}$. To do this, note that

$$[PQR] = \frac{1}{2}\left|\det\begin{pmatrix} 0 & F_{n+2} & 1 \\ F_{n+3} & F_n & 1 \\ F_{n+4} & 0 & 1 \end{pmatrix}\right|$$

so

$$\begin{aligned} 2[PQR] &= |F_{n+2}F_{n+4} - F_{n+2}F_{n+3} - F_nF_{n+4}| \\ &= |F_{n+2}(F_{n+4} - F_{n+3}) - F_nF_{n+4}| \\ &= |F_{n+2}^2 - F_nF_{n+4}|. \end{aligned}$$

Alternatively, if R_x and R_y are the projections of R on the x and y axes, respectively, and O is the origin,

$$\begin{aligned} [PQR] &= [PR_yR] + [QR_xR] + [OR_xRR_y] - [R_xOR_y] \\ 2[PQR] &= F_{n+3}(F_{n+2} - F_n) + F_n(F_{n+4} - F_{n+3}) + 2F_{n+3}F_n - F_{n+4}F_{n+2} \\ &= F_{n+4}F_n - F_{n+2}(F_{n+4} - F_{n+3}) = F_{n+4}F_n - F_{n+2}^2 \end{aligned}$$

This is the "same" as the area found above. Further use of the Fibonnaci definition yields

$$\begin{aligned} 2[PQR] &= (F_{n+2} + F_{n+3})(F_{n+2} - F_{n+1}) - F_{n+2}^2 = F_{n+2}(F_{n+3} - F_{n+1}) - F_{n+3}F_{n+1} \\ &= F_{n+2}^2 - F_{n+3}F_{n+1} \end{aligned}$$

By Cassini's Identity, $F_{k-1}F_{k+1} - F_k^2 = (-1)^k$, and so $[PQR] = \frac{1}{2}|(-1)^k| = \frac{1}{2}$.
Alternatively,

$$\begin{aligned} F_{n+2}^2 - F_{n+3}F_{n+1} &= F_{n+2}^2 - (F_{n+2} + F_{n+1})F_{n+1} = F_{n+2}(F_{n+2} - F_{n+1}) - F_{n+1}^2 \\ &= F_{n+2}F_n - F_{n+1}^2 \end{aligned}$$

and so

$$\begin{aligned} 2[PQR] &= |F_{n+2}^2 - F_{n+3}F_{n+1}| = |F_{n+1}^2 - F_{n+2}F_n| = |F_n^2 - F_{n+1}F_{n-1}| \\ &= \cdots = |F_2^2 - F_3F_1| = 1. \end{aligned}$$

Therefore, $[PQR] = \frac{1}{2}$. $\square$

The winners of the PEM Olympiad 2016 are presented in the next page. For pictures from the awarding and closing ceremony of PEM, please visit the Facebook page.

PEM Olympiad 2016 Winners

Level A

Bronze

Lorenzo Barzaga (De La Salle Santiago Zobel)
Andrew Demition (Valenzuela School of Mathematics and Science)
Allen Ross Mercado (Valenzuela School of Mathematics and Science)
Seth Gabriel Ricafort (Philippine Science High School)

Levels B and C

Gold

Shaquille Wyan Que (Grace Christian College)
Farrell Eldrian Wu (MGC New Life Christian Academy)

Silver

Alfonso Miguel Abella (Grace Christian College)
Brian Godwin Lim (Chiang Kai Shek College)
Dion Stephan Ong (Ateneo de Manila Junior HS)

Bronze

Jayson Dwight Catindig (Ateneo de Manila Senior HS)
Adam Christopher Chan (Grace Christian College)
Rodrigo Dexter Perando (Philippine Science HS)
Joseph Rodelas (Manila Science HS)
Rafael Santiago (Philippine Science HS)
Sean Marcus So (Grace Christian College)
Genesis Jacinth Tan (Quezon City Science HS)
Lance Ricco Teng (Makati Hope Christian School)

About the Program of Excellence in Mathematics

The Program of Excellence in Mathematics (PEM) is a yearly enrichment program in Mathematical Problem Solving started in 1989. Originally established by the late Dr. Jose Marasigan to train the countrys representatives to the International Mathematical Olympiad (IMO), it aims to discover, encourage, and challenge gifted students by providing them with a special program of instruction in various fields of mathematics. Topics include number theory, combinatorics, advanced Euclidean geometry, algorithms and computing.

At the core of PEMs objectives is the development of problem solving skills. The program exposes its participants to mathematical challenges and provides them with experiences that will improve the way they handle and solve problems in mathematics.

About Tuklás Matemátika

Tuklás Matemátika is the online journal of the Ateneo Mathematics Department. Aimed at exposing high school students and the general public to topics beyond usual high school curricula, Tuklás features informative and interesting articles about people and ideas from different areas of mathematics. Tuklás Matemátika originally served as an online supplementary journal for the PEM.

We encourage our readers to email us (`ateneo.tuklas@gmail.com`) or post on our Facebook page for questions, comments, and topic requests.