PROGRAM OF EXCELLENCE IN MATHEMATICS — Weekly Problems

Week 1 Solutions

1. Zach discovers in the library that a book has some number of consecutive pages torn out. Find the page numbers of the missing pages given that their sum is 452 and the print is double-sided.

   Solution. Let n be the number of missing pages, and assume that these have page numbers $a, a+1, ..., a+n-1$. Then $a+(a+1)+...+(a+n-1) = 452$. This means that the average page number of the missing pages is $\frac{452}{n}$. As consecutive integers lie symmetrically with respect to their arithmetic mean, we must have $\frac{452}{n} = \frac{a+(a+n-1)}{2}$, which implies that $n \cdot (2a+n-1) = 904 = 2^3 \cdot 113$. As the number n of missing pages is even, $2a+n-1$ is odd.

   The odd divisors of 904 are 1 and 113, but the case $2a+n-1 = 1$ is impossible since a must be positive. Thus, $2a+n-1 = 113$ and $n = 8$. Solving the equation $2a+7 = 113$ gives $a = 53$. Therefore, the page numbers of the pages torn out are $\boxed{\text{53 through 60}}$. □

   Solved by Alfonso Miguel Abella (Grace Christian College), Stefanie Alabastro (Assumption College San Lorenzo), Clyde Wesley Ang (Chiang Kai Shek College), Dan Baterisna (Colegio San Agustin-Makati), Jayson Dwight Catindig (Ateneo de Manila Senior HS), Kian Rey Chua (Quezon City Science HS), Klarisse Go (Immaculate Conception Academy (Greenhills)), Hye Yeon Kim (Immaculate Conception Academy (Greenhills)), Grant King (Grace Christian College), Brian Godwin Lim (Chiang Kai Shek College), Allen Ross Mercado (Valenzuela School of Math and Science), Dion Stephan Ong (Ateneo de Manila Junior HS), Gerard Francis Ortega (Ateneo de Manila Senior HS), Eryka Panganiban (Quezon City Science HS), Carl Joshua Quines (Valenzuela School of Math and Science), Julia Rafallo (Assumption Antipolo), Genesis Jacinth Tan (Quezon City Science HS), Justin Tan (Quezon City Science HS), Madeline Tee (Jubilee Christian Academy), Terence Tsai (Chiang Kai Shek College), Farrell Wu (MGC New Life Christian Academy).

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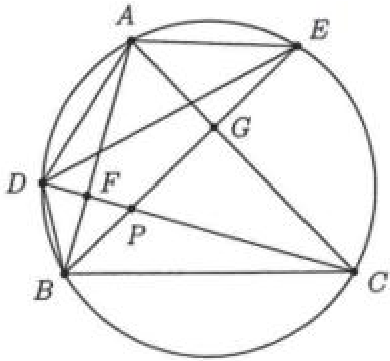
2. In ΔABC, $\angle BAC$ is less than $90°$. The perpendiculars from C on AB and from B on AC intersect the circumcircle of ΔABC again at D and E respectively. If $|DE| = |BC|$, determine the measure of $\angle BAC$.

   Solution. Let CD and BE intersect AB and AC at F and G respectively. Let P be the intersection point of CD and BE.

   Since $\angle DEB = \angle DCB$ and $|DE| = |BC|$, it follows that ΔDPE is congruent to ΔBPC. In particular, $|DP| = |BP|$. This implies that $\angle BDP = \angle PBD$.

   The quadrilateral $FPGA$ is cyclic as it has right angles at both F and G. Hence, $\angle BAC = \angle DPB$. Finally, $\angle BDP = \angle BDC = \angle BAC$ as both are subtended by BC. Hence, ΔBDP is equilateral. Therefore, $\angle BAC = \angle DPB = \boxed{60°}$. □



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3. The number A is composed of 3^{2016} 3s and no other digits. Find the highest power of 3 which divides A.

Solution. We consider more generally the number A_n, which is composed of 3^n digit 3s. Then, the number $B_n = \frac{1}{3}A_n$ consists of 3^n digit 1s.

Let M_n be the number which is formed of a digit 1, $(3^n - 1)$ consecutive digits 0, another digit 1, another $(3^n - 1)$ consecutive digits 0, and then another 1. Notice that $B_n \cdot M_n = B_{n+1}$.

Since the sum of the digits of M_n is 3, it is divisible by 3. However, since this is not divisible by 9, M_n is not divisible by 9. So B_{n+1} is divisible by only one higher power of 3 than B_n.

Now $B_1 = 111$ is divisible by 3^1 but not by 3^2, and so B_n is divisible by 3^n but not by 3^{n+1}. Thus, A_n is divisible by 3^{n+1} but not by 3^{n+2}. For our given number (where $n = 2016$), this means that the highest power that divides must be $\boxed{3^{2017}}$. □

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Weekly Problems

About the PEM Weekly Problems

The PEM Weekly Problems aims to challenge and enrich high school students' creativity and critical thinking skills by exposing them to non-routine math problems and puzzles. While the problem sets are primarily intended for PEM participants, everyone is encouraged to submit their solutions to us. We acknowledge on the page everyone who submits correct answers. Moreover, PEM participants who solve the most number of problems will be recognized and awarded during the PEM closing ceremony.

For the latest set of problems, visit `www.ateneo.edu/ls/sose/mathematics/pem-weekly-problems`.

Submitting Solutions

1. Typeset and handwritten solutions are welcome. For handwritten solutions, please scan or take a clear photo of your paper.
2. Indicate in the submission your name, school, and year level.
3. Send your solution to `ateneo.tuklas@gmail.com`.