Week 2 Solutions

4. An urn contains 20 balls colored either red, white, or black. If the number of black balls in the urn is doubled, the probability of drawing a red ball becomes 1/25 less than the probability of drawing a red ball in the beginning. If we remove all red balls, the probability of drawing a black ball becomes 1/16 more than the probability of drawing a black ball in the beginning. How many white balls are there?

Solution. Let a, b, and c be the number of red, black, and white balls, respectively. We have the system

$$a+b+c = 20 \tag{0.1}$$
$$\frac{a}{20+b} = \frac{a}{20}-\frac{1}{25} \tag{0.2}$$
$$\frac{b}{20-a} = \frac{b}{20}+\frac{1}{16} \tag{0.3}$$

Dividing the above two equations gives

$$\frac{20-a}{20+b}=\frac{16}{25} \Leftrightarrow b=\frac{180-25a}{16}$$

Substituting to (3) gives

$$\frac{180a-25a^2}{16}=\frac{100-5a}{4},$$

which has solution $a=4$. Thus, $b=5$ and $c=20-a-b=\boxed{11}$. □

Solved by Alfonso Miguel Abella (Grace Christian College), Aeram Clester Albo (Quezon City Science HS), Clyde Wesley Ang (Chiang Kai Shek College), Dan Alden Baterisna (Colegio San Agustin-Makati), Jayson Dwight Catindig (Ateneo de Manila Senior HS), Kian Rey Chua (Quezon City Science HS), Andrew Demition (Valenzuela School of Math and Science), Sophia Dominique Dizon (Ateneo de Manila Senior HS), Grant King (Grace Christian College), Brian Godwin Lim (Chiang Kai Shek College), Allen Ross Mercado (Valenzuela School of Math and Science), Dion Stephan Ong (Ateneo de Manila Senior HS), Eryka Panganiban (Quezon City Science HS), Shaquille Wyan Que (Grace Christian College), Carl Joshua Quines (Valenzuela School of Math and Science), Joseph Rodelas (Manila Science HS) Cylene Sabio (St. Scholastica's College, Manila) Genesis Jacinth Tan (Quezon City Science HS), Madeline Tee (Jubilee Christian Academy), Terence Tsai (Chiang Kai Shek College), and Farrell Wu (MGC New Life Christian Academy).

Partial credit to Stefanie Alabastro (Assumption College San Lorenzo), Adam Christopher Chan (Grace Christian College), Andrew Demition (Valenzuela School of Math and Science), Hye Yeon Kim (Immaculate Conception Academy (Greenhills)), Hyomin Lee (Lourdes School of Mandaluyong), and Justin Tan (Quezon City Science HS).

5. Find the largest positive integer $n < 2016$ for which the sum of the reciprocals of the non-zero digits of the integers from 1 to 10^n (inclusive) is also an integer.

   Solution. Denote by S_n the sum of reciprocals of the nonzero digits from 1 to 10^n, and let $K = \sum_{i=1}^{9} \frac{1}{i}$. Examining the terms in S_1, we see that $S_1 = K + 1$ since each digit n appears once and 1 appears an extra time. Now consider writing out S_2. Each term of K will appear 10 times in the units place and 10 times in the tens place (plus one extra 1 will appear), so $S_2 = 20K + 1$.

   In general, we will have that $S_n = (n10^{n-1})K + 1$ because each digit will appear 10^{n-1} times in each place in the numbers $1, 2, \ldots, 10^n - 1$, and there are n total places.

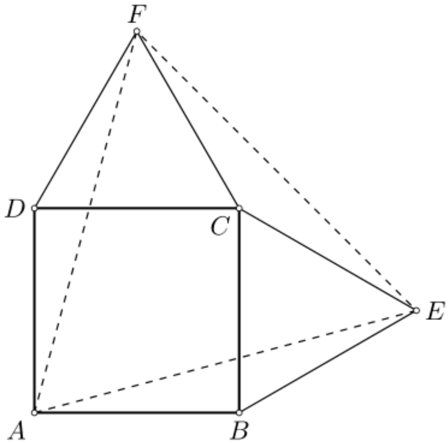
   The denominator of K is $D = 2^3 \cdot 3^2 \cdot 5 \cdot 7$. For S_n to be an integer, $n10^{n-1}$ must be divisible by D. Since 10^{n-1} only contains the factors 2 and 5 (but will contain enough of them when $n \geq 3$), we must choose n to be divisible by $3^2 \cdot 7$. As 2016 is a multiple of 63, the largest such n which is less than 2016 is $2016 - 63 = \boxed{1953}$. □

   Solved by Alfonso Miguel Abella (Grace Christian College), Clyde Wesley Ang (Chiang Kai Shek College), Dan Alden Baterisna (Colegio San Agustin-Makati), Jayson Dwight Catindig (Ateneo de Manila Senior HS), Adam Christopher Chan (Grace Christian College), Allen Ross Mercado (Valenzuela School of Math and Science), Dion Stephan Ong (Ateneo de Manila Junior HS), Carl Joshua Quines (Valenzuela School of Math and Science), Cylene Sabio (St. Scholastica's College Manila), Terence Tsai (Chiang Kai Shek College), and Farrell Wu (MGC New Life Christian Academy).

   Partial credit to Vincent Dela Cruz (Valenzuela School of Math and Science) and Justin Tan (Quezon City Science HS).

6. A square $ABCD$ and two points E and F outside of this square are given so that the triangles BEC and CFD are equilateral. Prove that the triangle AEF is also equilateral.

Solution. Since $ABCD$ is a square and the triangles BEC and CFD are equilateral, we have $|EB| = |BA| = |AD| = |DF| = |CE| = |CF|$. Thus the triangles EBA, ADF, and ECF are all isosceles. Also, $\angle EBA = \angle ADF = 90° + 60° = 150°$ and $\angle ECF = 360° - 90° - 2 \cdot 60° = 150°$. Therefore the triangles EBA, ADF, and ECF have two sides and the included angle congruent and are hence congruent themselves. It follows that $|EA| = |AF| = |EF|$ and thus triangle AEF is equilateral. □



Solved by Alfonso Miguel Abella (Grace Christian College), Aeram Clester Albo (Quezon City Science HS), Clyde Wesley Ang (Chiang Kai Shek College), Dan Alden Baterisna (Colegio San Agustin-Makati), Jayson Dwight Catindig (Ateneo de Manila Senior HS), Adam Christopher Chan (Grace Christian College), Kian Rey Chua (Quezon City Science HS), Mark Gabriel Danganan (Quezon City Science HS), Vincent Dela Cruz (Valenzuela School of Math and Science), Andrew Demition (Valenzuela School of Math and Science), Sophia Dominique Dizon (Ateneo de Manila Senior HS), Klarisse Go (Immaculate Conception Academy-Greenhills), Grant King (Grace Christian College), Brian Godwin Lim (Chiang Kai Shek College), Allen Ross Mercado (Valenzuela School of Math and Science), Dion Stephan Ong (Ateneo de Manila Senior HS), Eryka Panganiban (Quezon City Science HS), Shaquille Wyan Que (Grace Christian College), Carl Joshua Quines (Valenzuela School of Math and Science), Genesis Jacinth Tan (Quezon City Science HS), Madeline Tee (Jubilee Christian Academy), Terence Tsai (Chiang Kai Shek College), and Farrell Wu (MGC New Life Christian Academy).

Partial credit to Lucille Lacanienta (Assumption Antipolo) and Julianne Siccion (Assumption Antipolo).

7. Let ABC be an acute-angled triangle with altitude AD, and let M be any point on AD. Denote by E the intersection of the extension of BM with AC, and F the intersection of the extension of CM with AB. Prove that $\angle ADE = \angle ADF$.

Solution. We consider a Cartesian system of coordinates with BC and AD as the x- and y-axes, respectively (the origin is at D). Let $A(0, a)$, $B(b, 0)$, $C(c, 0)$, $M(0, m)$. Because the triangle is acute, $a, c > 0$ and $b < 0$. Also, $m > 0$. The equation of BM is $mx + by = bm$, and the equation of AC is $ax + cy = ac$. Their intersection is

$$E\left(\frac{bc(a-m)}{ab-cm}, \frac{am(b-c)}{ab-cm}\right).$$

Note that the denominator is strictly negative, hence nonzero. The point E therefore exists.

The slope of the line DE is the ratio of the coordinates of E, namely,

$$\frac{am(b-c)}{bc(a-m)}.$$

Interchanging b and c, we find that the slope of DF is

$$\frac{am(c-b)}{bc(a-m)}$$

which is the negative of the slope of DE. It follows that the lines DE and DF are symmetric with respect to the y-axis, i.e., $\angle ADE$ and $\angle ADF$ are equal. □

(*Remark:* A few submissions have correctly noted that this problem is actually a result on concurrent cevians known as *Blanchet's Theorem*.)

Solved by Clyde Wesley Ang (Chiang Kai Shek College), Dan Alden Baterisna (Colegio San Agustin-Makati), Vincent Dela Cruz (Valenzuela School of Math and Science), Dion Stephan Ong (Ateneo de Manila Junior HS), Shaquille Wyan Que (Grace Christian College), Carl Joshua Quines (Valenzuela School of Math and Science), Terence Tsai (Chiang Kai Shek College), and Farrell Wu (MGC New Life Christian Academy).

8. If $x+y+z=0$, prove that
$$\frac{x^2+y^2+z^2}{2}\cdot\frac{x^5+y^5+z^5}{5}=\frac{x^7+y^7+z^7}{7}.$$

Solution. Consider the polynomial $P(X)=X^3+pX+q$, whose zeros are x,y,z. Then
$$x^2+y^2+z^2=(x+y+z)^2-2(xy+xz+yz)=-2p.$$
Adding the relations $x^3=-px-q$, $y^3=-py-q$, and z^3-pz-q, which hold since x,y,z are zeros of $P(X)$, we obtain
$$x^3+y^3+z^3=-3q.$$
Similarly,
$$x^4+y^4+z^4=-p(x^2+y^2+z^2)-q(x+y+z)=2p^2,$$
and therefore
$$x^5+y^5+z^5=-p(x^3+y^3+z^3)-q(x^2+y^2+z^2)=5pq,$$
$$x^7+y^7+z^7=-p(x^5+y^5+z^5)-q(x^4+y^4+z^4)=-5p^2q-2p^2q=-7p^2q.$$
The relation from the statement reduces to the obvious
$$\frac{-2p}{2}\cdot\frac{5pq}{5}=\frac{-7p^2q}{7}.$$ □

Solved by Clyde Wesley Ang (Chiang Kai Shek College), Dan Alden Baterisna (Colegio San Agustin-Makati), Vincent Dela Cruz (Valenzuela School of Math and Science), Sophia Dominique Dizon (Ateneo de Manila Senior HS), Brian Godwin Lim (Chiang Kai Shek College),' Dion Stephan Ong (Ateneo de Manila Junior HS), Carl Joshua Quines (Valenzuela School of Math and Science), Julianne Siccion (Assumption Antipolo), Justin Tan (Quezon City Science HS), Madeline Tee (Jubilee Christian Academy), Terence Tsai (Chiang Kai Shek College), and Farrell Wu (MGC New Life Christian Academy).

9. Determine the number of ordered pairs of positive integers (x,y) which satisfy the equation
$$\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{y}}=\frac{1}{\sqrt{2016}}$$

Solution. Since $\sqrt{2016} = 12\sqrt{14}$,

$$\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}} = \frac{1}{12\sqrt{14}} \Leftrightarrow \frac{1}{\sqrt{14x}} = \frac{1}{\sqrt{14y}} + \frac{1}{168}.$$

which yields

$$\frac{1}{14x} = \frac{1}{14y} + \frac{1}{168^2} + \frac{2}{168\sqrt{14y}}.$$

This implies that both $\sqrt{14y}$ and $\sqrt{14x}$ must be rational numbers. Expressing $x = 14a$ and $y = 14b$, where a and b are positive integers, we obtain $1/a - 1/b = 1/12$, and hence

$$a = \frac{12b}{b+12} = 12 - \frac{144}{b+12}.$$

This means that $b+12|144$. The possible values of b are $4, 6, 12, 24, 36, 60, 132$, which correspond to a being $3, 4, 6, 8, 9, 10, 11$, respectively. Thus, there are $\boxed{7}$ ordered pairs of positive integers which satisfy the equation. □

Solved by Alfonso Miguel Abella (Grace Christian College), Aeram Clester Albo (Quezon City Science HS), Clyde Wesley Ang (Chiang Kai Shek College), Dan Alden Baterisna (Colegio San Agustin-Makati), Vincent Dela Cruz (Valenzuela School of Math and Science), Dion Stephan Ong (Ateneo de Manila Junior HS), Joseph Rodelas (Manila Science HS), Genesis Jacinth Tan (Quezon City Science HS), Terence Tsai (Chiang Kai Shek College), and Farrell Wu (MGC New Life Christian Academy).

Partial credit to Brian Godwin Lim (Chiang Kai Shek College).

Problems Posted: Sep. 10, 2016 Solutions Posted: Oct. 9, 2016