PROGRAM OF EXCELLENCE IN MATHEMATICS

Weekly Problems

Week 3 Solutions

10. Find the exact value of x, where

$$x = \frac{\displaystyle\sum_{n=1}^{44} \sin n^\circ}{\displaystyle\sum_{n=1}^{44} \cos n^\circ}$$

Solution. Using the sum-to-product identities

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

we have

$$\begin{aligned} x &= \frac{\sum_{n=1}^{44} \sin n^\circ}{\sum_{n=1}^{44} \cos n^\circ} \\ &= \frac{(\sin 1^\circ + \sin 44^\circ) + (\sin 2^\circ + \sin 43^\circ) ... + (\sin 22^\circ + \sin 23^\circ)}{(\cos 1^\circ + \cos 44^\circ) + (\cos 2^\circ + \cos 43^\circ) ... + (\cos 22^\circ + \cos 23^\circ)} \\ &= \frac{2\sin \frac{45}{2}^\circ (\cos \frac{43}{2}^\circ + \cos \frac{41}{2}^\circ + ... + \cos \frac{1}{2}^\circ)}{2\cos \frac{45}{2}^\circ (\cos \frac{43}{2}^\circ + \cos \frac{41}{2}^\circ + ... + \cos \frac{1}{2}^\circ)} \\ &= \tan \frac{45}{2}^\circ \\ &= \frac{\sin 45^\circ}{1 + \cos 45^\circ} \end{aligned}$$

Simplifying the expression in the previous line, we obtain $x = \boxed{\sqrt{2} - 1}$. $\square$

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11. Let X and Y be two points which lie on the arc BC of the circumcircle of ΔABC such that $\angle BAX = \angle CAY$. If M is the midpoint of the chord AX, prove that $BM + CM > AY$.

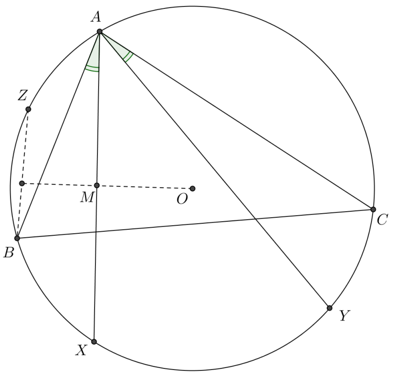
Solution. Let O be the circumcenter of ΔABC. Then $OM \perp AX$. Construct a perpendicular line from B to the extension of OM, and let this line intersect the circumcircle at Z. Since $OM \perp BZ$, OM is the perpendicular bisector of BZ. This means that $MZ = MB$. By the triangle inequality, we have

$$BM + MC = ZM + MC > CZ.$$

But $BZ || AX$, thus

$$\overset{\frown}{AZ} = \overset{\frown}{BX} = \overset{\frown}{CY} \implies Z\overset{\frown}{AC} = Y\overset{\frown}{CA} \implies CZ = AY.$$

It then follows that $BM + CM > AY$. ☐



Solved by Clyde Wesley Ang (Chiang Kai Shek College), Allen Ross Mercado (Valenzuela School of Math and Science), Dion Stephan Ong (Ateneo de Manila Senior HS), Shaquille Wyan Que (Grace Christian College), Carl Joshua Quines (Valenzuela School of Math and Science), and Farrell Wu (MGC New Life Christian Academy).

Partial credit to Madeline Tee (Jubilee Christian Academy)

12. Let A be a set containing 2016 real numbers. Define

$$S = \{x + y | x, y \in A\}, D = \{x - y | x, y \in A\}.$$

Prove that $|A| \cdot |D| \leq |S|^2$. (Here, $|X|$ refers to the cardinality of the set X.)

Solution. We prove this more generally for any finite set A of real numbers. Define the function $f : A \times D \to S \times S$ in the following manner: for $a \in A$ and $d \in D$, let

$$f(a, d) = (a + x_d, a + y_d),$$

where $x_d, y_d \in A$ are chosen such that $x_d - y_d = d$ and x_d is maximal. Clearly, for a given d, x_d and y_d exist and are uniquely determined.

If $f(a, d) = f(a', d')$ then $a + x_d = a' + x_{d'}$ and $a + y_d = a' + y_{d'}$. It follows that $x_d - y_d = x_{d'} - y_{d'}$. Therefore, $d = d'$ and hence $a = a'$ as well.

Thus, f is one-to-one, and $|A \times D| \leq |S \times S|$, so $|A| \cdot |D| \leq |S|^2$. □

Solved by Alfonso Miguel Abella (Grace Christian College) and Farrell Wu (MGC New Life Christian Academy).

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About the PEM Weekly Problems

The PEM Weekly Problems aims to challenge and enrich high school students' creativity and critical thinking skills by exposing them to non-routine math problems and puzzles. While the problem sets are primarily intended for PEM participants, everyone is encouraged to submit their solutions to us. We acknowledge on the page everyone who submits correct answers. Moreover, PEM participants who solve the most number of problems will be recognized and awarded during the PEM closing ceremony.

For the latest set of problems, visit `www.ateneo.edu/ls/sose/mathematics/pem-weekly-problems`.

Submitting Solutions

1. Typeset and handwritten solutions are welcome. For handwritten solutions, please scan or take a clear photo of your paper.
2. Indicate in the submission your name, school, and year level.
3. Send your solution to `ateneo.tuklas@gmail.com`.