

Week 5 Solutions

16. Let a and b be real numbers such that $\frac{a^2}{1+a^2} + \frac{b^2}{1+b^2} = 1$. Determine all possible values of the expression

$$(a+b) \left(\frac{a}{1+a^2} + \frac{b}{1+b^2} \right).$$

Solution. Multiplying the equation by $(1+a^2)(1+b^2)$ and simplifying gives $a^2b^2 = 1$. Therefore, $ab = \pm 1$. Now

$$\begin{aligned} (a+b) \left(\frac{a}{1+a^2} + \frac{b}{1+b^2} \right) &= (a+b) \cdot \frac{a+ab+b+a^2b}{(1+a^2)(1+b^2)} \\ &= (a+b) \cdot \frac{(a+b)(1+ab)}{(1+a^2)(1+b^2)} \\ &= \frac{(a^2+b^2+2ab)(1+ab)}{1+a^2+b^2+a^2b^2}. \end{aligned}$$

If $ab = -1$, the value of this expression equals 0, while if $ab = 1$, this expression equals 2. Thus, the only possible values are 0 and 2. \square

(*Note:* We can also show that these two values are actually achieved. For example, the value 0 is attained when $a = 1$ and $b = -1$, and the value 2 is attained when $a = b = 1$.)

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17. Let m and n be integers such that $1 \leq m \leq n$. Prove that m divides the number

$$n \sum_{k=0}^{m-1} (-1)^k \binom{n}{k}.$$

Solution. Note that

$$\begin{aligned}
 n \sum_{k=0}^{m-1} (-1)^k \binom{n}{k} &= n \sum_{k=0}^{m-1} (-1)^k \left(\binom{n-1}{k} + \binom{n-1}{k-1} \right) \\
 &= n \sum_{k=0}^{m-1} (-1)^k \binom{n-1}{k} - n \sum_{k=0}^{m-2} (-1)^k \binom{n-1}{k} \\
 &= n(-1)^{m-1} \binom{n-1}{m-1} \\
 &= m(-1)^{m-1} \binom{n}{m},
 \end{aligned}$$

which is clearly divisible by m . □

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Partial credit to Madeline Tee (Jubilee Christian Academy) and Terence Tsai (Chiang Kai Shek College).

18. Let A be a subset of the set $\{1, 2, 3, \dots, 15\}$ such that no three distinct integers in A have a product which is a perfect square. What is the maximum number of elements that A can have?

Solution. Note that the product of the three elements in each of the sets $\{1, 4, 9\}$, $\{2, 6, 12\}$, $\{3, 5, 15\}$, and $\{7, 8, 14\}$ is a square. Hence, none of these sets can be a subset of A . Because they are disjoint, it follows that A has at most 11 elements.

Since 10 is not an element of the aforementioned sets, if $10 \notin A$, then A has at most 10 elements. Suppose $10 \in A$. Then none of $\{2, 5\}$, $\{6, 15\}$, $\{1, 4, 9\}$, and $\{7, 8, 14\}$ is a subset of A . If $\{3, 12\} \not\subset A$, it follows that A has at most 10 elements. If $\{3, 12\} \subset A$, then none of $\{1\}$, $\{4\}$, $\{9\}$, $\{2, 6\}$, $\{5, 15\}$, and $\{7, 8, 14\}$ is a subset of A , and then A has at most 9 elements. We conclude that A has at most 10 elements in any case.

Finally, it is easy to verify that the subset

$$A = \{1, 4, 5, 6, 7, 10, 11, 12, 13, 14\}$$

has the desired property. Hence, the maximum number of elements of A is 10. \square

(*Note:* This is just a re-worded version of a problem proposed by Bulgaria which appeared in the shortlist to the 1994 IMO.)

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The PEM Weekly Problems aims to challenge and enrich high school students' creativity and critical thinking skills by exposing them to non-routine math problems and puzzles. While the problem sets are primarily intended for PEM participants, everyone is encouraged to submit their solutions to us. We acknowledge on the page everyone who submits correct answers. Moreover, PEM participants who solve the most number of problems will be recognized and awarded during the PEM closing ceremony.

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1. Typeset and handwritten solutions are welcome. For handwritten solutions, please scan or take a clear photo of your paper.
2. Indicate in the submission your name, school, and year level.
3. Send your solution to ateneo.tuklas@gmail.com.