Week 5 Solutions

16. Let a and b be real numbers such that $\frac{a^2}{1+a^2} + \frac{b^2}{1+b^2} = 1$. Determine all possible values of the expression

$$(a+b)\left(\frac{a}{1+a^2} + \frac{b}{1+b^2}\right).$$

Solution. Multiplying the equation by $(1+a^2)(1+b^2)$ and simplifying gives $a^2b^2 = 1$. Therefore, $ab = \pm 1$. Now

$$\begin{aligned}(a+b)\left(\frac{a}{1+a^2} + \frac{b}{1+b^2}\right) &= (a+b)\cdot\frac{a+ab+b+a^2b}{(1+a^2)(1+b^2)} \\ &= (a+b)\cdot\frac{(a+b)(1+ab)}{(1+a^2)(1+b^2)} \\ &= \frac{(a^2+b^2+2ab)(1+ab)}{1+a^2+b^2+a^2b^2}.\end{aligned}$$

If $ab = -1$, the value of this expression equals 0, while if $ab = 1$, this expression equals 2. Thus, the only possible values are $\boxed{0 \text{ and } 2}$. □

(*Note:* We can also show that these two values are actually achieved. For example, the value 0 is attained when $a = 1$ and $b = -1$, and the value 2 is attained when $a = b = 1$.)

Solved by Clyde Wesley Ang (Chiang Kai Shek College), Dan Alden Baterisna (Colegio San Agustin-Makati), Jayson Dwight Catindig (Ateneo de Manila Senior HS), Brian Godwin Lim (Chiang Kai Shek College), Allen Ross Mercado (Valenzuela School of Math and Science), Dion Stephan Ong (Ateneo de Manila Junior HS), Joseph Rodelas (Manila Science HS), Genesis Jacinth Tan (Quezon City Science HS), Madeline Tee (Jubilee Christian Academy), Terence Tsai (Chiang Kai Shek College), and Farrell Eldrian Wu (MGC New Life Christian Academy).

17. Let m and n be integers such that $1 \leq m \leq n$. Prove that m divides the number

$$n\sum_{k=0}^{m-1}(-1)^k\binom{n}{k}.$$

Solution. Note that

$$\begin{aligned} n\sum_{k=0}^{m-1}(-1)^k\binom{n}{k} &= n\sum_{k=0}^{m-1}(-1)^k\left(\binom{n-1}{k}+\binom{n-1}{k-1}\right) \\ &= n\sum_{k=0}^{m-1}(-1)^k\binom{n-1}{k} - n\sum_{k=0}^{m-2}(-1)^k\binom{n-1}{k} \\ &= n(-1)^{m-1}\binom{n-1}{m-1} \\ &= m(-1)^{m-1}\binom{n}{m}, \end{aligned}$$

which is clearly divisible by m. □

Solved by Clyde Wesley Ang (Chiang Kai Shek College), Dan Alden Baterisna (Colegio San Agustin-Makati), Dion Stephan Ong (Ateneo de Manila Junior HS), Carl Joshua Quines (Valenzuela School of Math and Science), and Farrell Eldrian Wu (MGC New Life Christian Academy).

Partial credit to Madeline Tee (Jubilee Christian Academy) and Terence Tsai (Chiang Kai Shek College).

18. Let A be a subset of the set $\{1, 2, 3, ..., 15\}$ such that no three distinct integers in A have a product which is a perfect square. What is the maximum number of elements that A can have?

Solution. Note that the product of the three elements in each of the sets $\{1, 4, 9\}$, $\{2, 6, 12\}$, $\{3, 5, 15\}$, and $\{7, 8, 14\}$ is a square. Hence, none of these sets can be a subset of A. Because they are disjoint, it follows that A has at most 11 elements.

Since 10 is not an element of the aforementioned sets, if $10 \notin A$, then A has at most 10 elements. Suppose $10 \in A$. Then none of $\{2, 5\}$, $\{6, 15\}$, $\{1, 4, 9\}$, and $\{7, 8, 14\}$ is a subset of A. If $\{3, 12\} \not\subset A$, it follows that A has at most 10 elements. If $\{3, 12\} \subset A$, then none of $\{1\}$, $\{4\}$, $\{9\}$, $\{2, 6\}$, $\{5, 15\}$, and $\{7, 8, 14\}$ is a subset of A, and then A has at most 9 elements. We conclude that A has at most 10 elements in any case.

Finally, it is easy to verify that the subset

$$A = \{1, 4, 5, 6, 7, 10, 11, 12, 13, 14\}$$

has the desired property. Hence, the maximum number of elements of A is 10. □

(*Note:* This is just a re-worded version of a problem proposed by Bulgaria which appeared in the shortlist to the 1994 IMO.)

Solved by Clyde Wesley Ang (Chiang Kai Shek College) and Dion Stephan Ong (Ateneo de Manila Junior HS).

Partial credit to Dan Alden Baterisna (Colegio San Agustin-Makati) and Farrell Eldrian Wu (MGC New Life Christian Academy)

Problems Posted: Oct. 16, 2016 Solutions Posted: Oct. 31, 2016