Tuklás Matemátika

Ateneo Mathematics Department

Steiner Sets and Edge Steiner Sets in a Graph

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Posted September 10, 2016 at `http://www.ateneo.edu/ls/sose/mathematics/tuklas`

In Graph Theory, a graph is composed of a set of vertices and a set of edges. One may imagine the vertices as rubber balls of very small radius and the edges as very elastic and flexible rubber tubes of small diameter that are used to connect vertices. The following discussion will be limited to graphs without loops (edges that start and end at the same vertex) or multiple edges (allowing two vertices to be connected by more than one edge).

Formally, a graph G is denoted by a pair $(V(G), E(G))$ where $V(G)$ is the set of vertices of G and $E(G)$ is the set of edges of G.

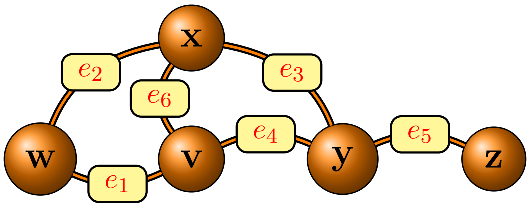

Figure 1: A graph with 5 vertices and 6 edges

Let G_1 be the graph in Figure 1. Then $V(G_1) = \{v, w, x, y, z\}$ and $E(G_1) = \{e_1, e_2, e_3, e_4, e_5, e_6\}$. Commonly, edges are denoted by writing down their end vertices; for example e_1 can be denoted by vw or wv. Also, $|V(G_1)| = 5$ and $|E(G_1)| = 6$. It can be shown that the maximum number of edges in a graph (without loops or multiple edges) of n vertices is $\frac{n(n-1)}{2}$.

Two vertices a and b are said to be *adjacent* if there is an edge with endpoints at a and b. Moreover, a vertex a is said to be *incident* to an edge e if a is one of the endpoints of e. For the graph in Figure 1, the vertex z is incident to e_5 and is adjacent to y.

An $a - b$ *path* in a graph G is a sequence of distinct edges that connect a sequence of distinct vertices starting at a and ending at b. For example, in Figure 1 there are two $x - v$ paths: $[x, w, v]$ and $[x, y, v]$. A path that starts and ends at the same vertex is called a *cycle*. In Figure 1, one cycle is $[y, v, w, x, y]$. A graph G is *connected* if there exists an $a - b$ path for each pair of vertices a, b in the set $V(G)$.

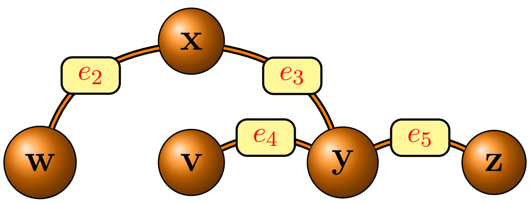

Figure 2: A graph with 5 vertices and 4 edges

A tree graph, or simply a *tree*, is a connected graph with no cycles. The graph in Figure 1 is not a tree. It can be observed that the graph in Figure 2 is connected and has no cycles, hence it is a tree.

Consider a subset W of $V(G)$, a Steiner W-tree T in G is a tree containing the vertices in W in such a way that $|V(T)|$ is minimum. In Figure 1, let the given graph be G_1 and $W_1 = \{w, y\}$. Then, there are two Steiner W_1-trees of G_1. These are the graphs given in Figure 3.

Figure 3: Steiner W_1-trees of the graph in Figure 1

Let $S(W)$ be the collection of vertices found in any Steiner W-tree of a graph G. If $S(W) = V(G)$, then W is called a *Steiner set* in G. Obviously, $V(G)$ is always a Steiner set in G. Let G_1 be the graph in Figure 1, $S(W_1) = S(\{w, y\}) = \{w, v, x, y\} \neq V(G_1)$. So W_1 is not a Steiner set in G_1. But if we consider $W_2 = \{w, z\}$, $S(W_2) = V(G_1)$ (Hint: Determine the Steiner W_2-trees of G_1). Hence, W_2 is a Steiner set in G_1. In fact, we cannot find a subset of $V(G_1)$ of lesser cardinality having this property. Consequently, W_2 is a minimum Steiner set in G_1.

Now, let $S_E(W)$ be the collection of edges found in any Steiner W-tree of a graph G. If $S_E(W) = E(G)$, then W is called an *edge Steiner set* in G. Clearly, $V(G)$ is always an edge Steiner set in G. It can be shown that any edge Steiner set is always a Steiner set. In the graph G_1 in Figure 1, $W_1 = \{w, y\}$ is not an edge Steiner set. Moreover, it can be verified that $S_E(W_2) = S_E(\{w, z\})$ does not contain the edge e_6 so $S_E(W_2) \neq E(G_1)$; thus, W_2 is also not an edge Steiner set of G_1. It can be verified that the set $W_3 = \{w, x, v, z\}$ is an edge Steiner set of G_1 and in fact is a minimum edge Steiner set.

The minimum cardinality among all Steiner sets of a graph G is called the *Steiner number* of G while the minimum cardinality among all edge Steiner sets of G is the *edge Steiner number* of G. The graph in Figure 1 has Steiner number 2 and edge Steiner number 4. Note that for any tree T, the edge Steiner number and the Steiner number coincide, and is equal to the number of vertices of the tree incident to exactly one edge. Hence, the Steiner number and edge Steiner number of the graph in Figure 2 is 3. More results on Steiner sets and edge Steiner sets are found in [1, 2, 3, 5, 6].

The problem of finding a Steiner set (or an edge Steiner set) is one of the many variations of the Steiner tree problem. The Steiner tree problem, named in honor of Jakob Steiner, is a combinatorial problem in finding the shortest interconnection possible given a set of objects. It has applications in circuit layout and network design.

References

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